Int. J. Solids Structures, 1970, Vol. 6, pp. 1323 to 1340. Pergamon Press. Printed in Great Britain

FORMULATION OF A STATISTICAL EQUATION OF MOTION OF A VISCOUS FLUID IN AN ANISOTROPIC NON-RIGID POROUS SOLID

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Abstract—The motion of a viscous fluid through an anisotropic non-rigid porous solid is studied. The porous solid considered in this study is a solid medium involving a very dense and fine network of tubules, whose diameter is much smaller than the characteristic length of the flow system. Kinematically, such a medium should behave like a continuum, i.e. the state of deformation can be well described macroscopically. Dynamically, this type of medium has to be distinguished from the true solid continuum, because one additional body force in addition to its weight (the surface traction exerted by the viscous fluid moving through it) has to be considered in the deformation of the porous solid. Their own weight is the only body force in a true solid continuum. The equations governing the macroscopic motion of the viscous fluid flow through the porous medium of this type are derived by averaging the motion of the fluid through individual elements of pores over a small volume of the porous medium. The physics of the derived equations and the possible applications to the blood flow in the network of capillary blood vessels and to the motion of extracellular fluid through the network of interstitial space are discussed.

NOTATION

a	radius of tubules composing the pore network
C	absorbed fluid content in the solid
f	pore matrix function
$\mathbf{f}$	density of the net stress force
$\mathbf{g}$	gravitational acceleration
$\mathscr{H}$	weighting tensor
$\mathscr{I}$	idemfactor
K	scalar permeability
$\mathscr{k}$	permeability tensor
$\mathscr{L}$	weighting tensor in effective stress in the solid
$\hat{n}$	unit normal vector of an area element
$\mathrm{d}n$	differential solid angle element about $\hat{n}$
N	hemispherical number density of pores
p	pressure
$\langle p \rangle$	macroscopic value of the pressure
$\hat{r}$	unit vector in the radial direction in a tubule
S	surface composed of solid–fluid interface within T

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$\mathscr{S}$ stress tensor for the solid constituent
t time
$\mathbf{t}$ stress
T material volume associated with the solid constituent
$T_{s,(f)}$ solid (fluid) portion of T
U axial component of the fluid velocity in a tubule composing the pore network
U_0 average value of U over the cross section of the tubule
$\mathbf{v}$ velocity
$\mathbf{V}_f$ microscopic value of the fluid velocity
$\mathbf{x}$ position vector
δ radius of the small spherical volume τ
λ correction factor for branching and joining flow
μ shear viscosity of the fluid
ρ density
$\mathscr{P}$ density tensor
σ surface area of small spherical volume τ
Σ surface area of the material volume associated with the solid constituent
$\Sigma_s, (\Sigma_f)$ solid (fluid) portion of Σ
$\mathrm{d}\sigma$ differential area element
τ small spherical volume over which the microscopic value is averaged
$\mathrm{d}\tau$ differential volume element
φ volume fraction of the fluid occupying the void space of the pore network
$\hat{\omega}, \hat{\Omega},$ unit vector designating the orientation in space
$\omega_{\hat{n}}, (\Omega_{\hat{\omega}})$ solid angle about $\hat{n}$. $(\hat{\omega})$
$\mathrm{d}\omega, \mathrm{d}\Omega$ differential solid angle element

$\frac{\partial}{\partial t}$ partial time derivative holding the position in space fixed

$\frac{\mathrm{d}}{\mathrm{d}t}$ partial time derivative holding the material surface associated with the solid constituent fixed

∇ gradient operator
∇^2 Laplace operator
. dot product
: double-dot product
$Q_s(Q_f)$ Q of solid (fluid) or solid (fluid) portion of Q
Q_0 constant value of Q.

1. INTRODUCTION

AN EQUATION governing the macroscopic motion of a fluid through an isotropic rigid porous medium was formulated a long time ago by Darcy [1] on empirical basis, and is known as Darcy's law:

$$\langle \mathbf{v}_f \rangle = -\frac{K}{\mu}(\nabla \langle p \rangle - \rho_f \mathbf{g}), \tag{1}$$

where $\langle \mathbf{v}_f \rangle$ is the macroscopic velocity of the fluid, ρ_f is the density of the fluid, μ is the shear viscosity of the fluid, $\langle p \rangle$ is the macroscopic fluid pressure, $\mathbf{g}$ is the gravitational acceleration, ∇ is the gradient operator and K is a constant called the *permeability* of the porous medium. This equation applies to the slow motion of a fluid through an isotropic porous solid, in which the distribution of the pore density has no preferred direction. For an anisotropic medium, such as a layered material, it has been suggested that K be regarded as a symmetric tensor, called the *permeability tensor* [2 p. 88].

The present writers are concerned with the extension of the porous media theory to deal with the problem of fluid motion in animal tissues. Many tissues are composed of

cells which are separated by interstitial spaces filled with extracellular fluid. In some organs these interstitial spaces form a multi-channeled network which plays an intermediary role in the exchange of fluids, gases and ions between blood and cells [3]. As the tissue cells are deformable, the medium may be considered as a deformable porous medium. The concept of the flow through a deformable porous solid may also be useful to describe the motion of blood through a network of capillary blood vessels in muscle tissues.

In this study, we will limit our discussions to a specific type of porous solid; a solid medium which involves a very dense and fine network of tubules whose diameter is much smaller than the characteristic length of the flow system. It should be mentioned that the region occupied by the solid medium of this type material is a connected region and, consequently, it should behave like an ordinary solid continuum as far as the kinematics of deformation is concerned. Since the diameter of the tubules forming the pore network is supposedly very small, we assume that the macroscopic deformation of the medium is well defined everywhere. For example, we can prepare a solid continuum which has the same local deformation as the porous medium under consideration. If we carry out the stress–strain test of this porous medium in a vacuum chamber, we should be able to find the constitutive relation of this material. Such relation must represent the material property of the porous solid and enable us to find the state of deformation under the known state of loading and vice versa provided the pores remain void.

In this study, we assume that the constitutive relation of the porous medium under consideration is known and that we can treat the dynamics of the deformation of the non-rigid porous solid as a continuum by including an additional body force, the surface traction exerted by the viscous fluid moving through the tubules, in the equilibrium equation. One may question the value of any further analysis when the constitutive equation of the porous solid is known. For a deformable porous solid, the motion of the viscous fluid through it causes the deformation of medium, which changes the pore distribution and the diameter of the pore tubules. As a consequence, the deformation of the porous solid changes the fluid flow through it. In other words, the motion of the fluid and the porous solid are coupled together. The target of our study is to investigate the mechanism of the coupling of the motion of two constituents. The assumption that the constitutive equation of the porous solid is known is rather a practical one, because we can test a segment of tissue under normal state and isolate the material property (constitutive equation) of the porous solid by using the known coupling mechanism (which we are going to study). Then, we can analyze all possible state of tissue based on this information. As it will be shown, the material property of the porous solid appears in the equilibrium equation in an intact form. This is so because the porous solid occupies a connected region. In case of a porous medium composed of a fluid and many solid particles, the above feature is completely lacking because neither constituent is providing an intact bed for the motion of the remaining constituent.

In the literature [4–9] the compressibility of the fluid and the elasticity of the porous medium has been accounted for in the equation of continuity. Biot [10], Truesdell [11], Green and Naghdi [12, 13] and others have introduced generalized theories of mixtures and derived general equations of motion and constitutive equations. It is intuitively clear that the premeability of a porous medium is a function of the detailed structures of pores. Consequently, information on the detailed structure of pore is required for a theoretical determination of permeability. In general, an explicit description of the pore structure is rather difficult. In the present paper, we consider a special kind of porous

medium: (1) a connected solid medium having numerous circular cylindrical tubules of locally identical diameters; (2) the length of the individual segment of tubules is significantly larger than its radius; (3) the diameter of tubules is much smaller than the characteristic length of system; (4) the motion of the fluid relative to the porous solid is so slow that the inertia force due to the convection associated with the relative motion is negligible. Throughout this study, we use simply the "porous solid" for this specific medium unless otherwise mentioned.

In order to describe the pore distribution in the porous solid, we introduce the "pore matrix function" which is defined as the number of tubules per unit area of a plane per unit solid angle about the normal vector to the plane under consideration. The generalization of Darcy's law, the equation of continuity, and the macroscopic equations of motion of the porous solid are derived with respect to this pore matrix function. Our theory is more general and more realistic than some existing theories such as the parallel and serial type of straight capillary model by Scheidegger [14]. On the other hand, our study is less specific than the network model of Fatt [19] because we derive only a general form of the permeability tensor having an undetermined parameter. Our object is to derive specific relations between the physical characteristics of the constituents of the system and the physics of the flow in terms of the pore matrix function. A method parallel to this can be readily applied to pores of different geometry.

2. FORMULATION OF THE STATISTICAL EQUATION OF MOTION OF THE FLUID

The porous solid under consideration is a connected medium (like swiss cheese) having a fine pore network of tubules, whose diameter is much smaller than the characteristic length of the system. The characteristic length in this case should be the length associated with the change of the macroscopic velocity of the porous solid $\mathbf{v}_s$, which we designate by L. If we define $\mathbf{v}_s$ as the velocity of a solid continuum which undergoes the same deformation as the porous solid, then the true velocity of the porous solid is equal to $[\mathbf{v}_s+0(a/L)]$, where a designates the radius of the tubules forming the pore network. Since a/L is assumed negligibly small, $\mathbf{v}_s$ literally represents the motion of the solid medium of the porous solid under consideration. In other words, we can represent the *macroscopic motion* of the porous solid by $\mathbf{v}_s$ and then the true motion of the solid medium in the porous solid differs from $\mathbf{v}_s$ only by an order of magnitude of a/L, which is negligibly small.

It is convenient to represent the velocity of the fluid moving through individual tubules of the pore network as the sum of $\mathbf{v}_s$ and the velocity of the fluid relative to the porous solid:

$$\mathbf{V} = \mathbf{v}_s+\mathbf{V}_f, \tag{2}$$

where $\mathbf{v}_s$ is the macroscopic velocity of the porous solid whose space derivatives of various order are continuous, while $\mathbf{V}_f$ is the fluid velocity relative to the solid medium in a given tubule under observation. The macroscopic velocity of the fluid is equal to

$$\mathbf{v}_f = \mathbf{v}_s+\langle\mathbf{V}_f\rangle, \tag{3}$$

where $\langle Q\rangle$ designates the space averaged value of Q over small volume in space, whose

dimension is much greater than a, but is much smaller than L. The equation of motion of a Newtonian fluid is the Navier–Stokes equation [15]:

$$\rho_f\left(\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)=\rho_f\mathbf{g}-\nabla p+\mu\nabla^2\mathbf{V}, \tag{4}$$

where ρ_f is the density of the fluid, t is the time, $\mathbf{V}$ is the velocity of the fluid, $\mathbf{g}$ is the body force per unit mass, p is the pressure in the fluid, μ is the shear viscosity of the fluid and ∇ is the gradient operator. Since we assumed that a is very small, the resistance to the fluid flow has to be very large. Consequently, the magnitude of $\mathbf{V}_f$ has to be very small and the inertia force due to the time rate of change of $\mathbf{V}_f$ can be neglected. Thus, we replace $\mathbf{V}$ of the l.h.s. of (4) by $\mathbf{v}_s$. The characteristic length L is defined in connection with the space variation of $\mathbf{v}_s$ and, consequently, the order of magnitude of $\nabla^2\mathbf{v}_s$ is equal to $1/L^2$. The order of magnitude of $\nabla^2\mathbf{V}_f$ is equal to $1/a^2$. Since a is much smaller than L, we find that $\nabla^2\mathbf{v}_s$ is negligible compared to $\nabla^2\mathbf{V}_f$. Thus, we can replace $\mathbf{V}$ of the r.h.s. of (4) by $\mathbf{V}_f$. By doing so, we can write (4) as follows:

$$\rho_f\frac{\partial \mathbf{v}_s}{\partial t}=\rho_f\mathbf{g}-\nabla p+\mu\nabla^2\mathbf{V}_f, \tag{5}$$

where the inertia force term due to the convective acceleration of the porous solid medium is neglected because it remains very small in most cases.

Now, let us introduce a "pore matrix function" $f(\mathbf{x},\hat{\Omega})$ to describe the structure of the network of pores. This function should describe the number density and the orientation of the pores. We define $f(\mathbf{x},\hat{\Omega})$ statistically as the number of the pores per unit area per unit solid angle about an axis parallel to a unit vector $\hat{\Omega}$ on a plane normal to $\hat{\Omega}$ at a point $\mathbf{x}$ in space. It follows from this definition that the number of pores which are oriented within a small solid angle $d\Omega$ about $\hat{\Omega}$ and passing through a small element of area ΔS normal to a unit vector $\hat{\omega}$ is equal to

$$\Delta N(\mathbf{x},\hat{\Omega},\hat{\omega})=[f(\mathbf{x},\hat{\Omega})\hat{\Omega}\cdot\hat{\omega}\,d\Omega]\Delta S. \tag{6}$$

The total number of pores per unit area passing through an element area with unit normal vector $\hat{\omega}$ can be obtained by integrating equation (6) over a hemispherical solid angle about $\hat{\omega}$

$$N(\mathbf{x},\hat{\omega})=\int_{\Omega_{\hat{\omega}}=0}^{2\pi} f(\mathbf{x},\hat{\Omega})\hat{\Omega}\cdot\hat{\omega}\,d\Omega, \tag{7}$$

where $\Omega_{\hat{\omega}}$ is the solid angle about $\hat{\omega}$. We may call $N(\mathbf{x},\hat{\omega})$ the "hemispherical number density on the $\hat{\omega}$ plane at $\mathbf{x}$". For a point where a number of pores meet or branch, we should interpret $f(\mathbf{x},\hat{\Omega})$ as the pore matrix function associated with a hypothetical pore distribution which describes an equivalent passage of the fluid.

If we assume that the cross section of the pores are much smaller than the length of each segment of the pores, then, $\mathbf{V}_f$ in a pore in the direction of $\hat{\Omega}$ is more or less parallel to $\hat{\Omega}$ and, consequently, only the $\hat{\Omega}$-component of equation (5) need to be considered for the fluid movement through this pore. We write the $\hat{\Omega}$-component of the equation (5) in the following form:

$$\rho_f\left(\frac{\partial \mathbf{v}_s}{\partial t}-\mathbf{g}\right)\cdot\hat{\Omega}\hat{\Omega}=-\nabla p\cdot\hat{\Omega}\hat{\Omega}+\mu\nabla^2 U(\hat{\Omega})\hat{\Omega}, \tag{8}$$

where $U(\hat{\Omega})$ designates the magnitude of the velocity of the fluid moving through the pore with direction $\hat{\Omega}$. If the flow of the fluid in the pore is Poiseuillean, then we have $\nabla^2 U(\hat{\Omega}) = -8U_0(\hat{\Omega})/a^2$; in which $U_0(\hat{\Omega})$ represents the magnitude of the average velocity of the flow over the cross section of the pore and a is the radius of the pore. If the cross section of the pore is not a circle, then a should be interpreted as the hydraulic radius. Since each segment of pores forming the pore network has a finite length, the entry and exit effect must be considered. Therefore, we introduce a parameter λ associated with the entry and exit effect as follows:

$$\nabla^2 U(\hat{\Omega}) = \lambda\left[-8\frac{U_0(\hat{\Omega})}{a^2}\right]. \tag{9}$$

The value of λ should be a function of the geometry of the branching or joining of the different pores. Although some simple cases of branching flow (see Benner *et al.* [16] and Rose and Witherspoon [17]) and end effects (see Lew and Fung [18]) have been investigated, a general solution of the branching and joining flow is not available. Fortunately, the formulation of the governing equations of the macroscopic motion can be accomplished without introducing a specific information about λ. This is a rather fortunate circumstance because λ can be quite complex. In the following analysis, we assume that λ and a are independent of $\hat{\Omega}$. Substitution of (9) into (8) yields the equation

$$U_0(\hat{\Omega})\hat{\Omega}\hat{\Omega} = -\rho_f\left(\frac{\partial \mathbf{v}_s}{\partial t} - \mathbf{g}\right)\cdot\frac{a^2}{8\lambda\mu}\hat{\Omega}\hat{\Omega} - \nabla p\cdot\frac{a^2}{8\lambda\mu}\hat{\Omega}\hat{\Omega}\hat{\Omega}. \tag{10}$$

The macroscopic value of the velocity of the fluid relative to the porous solid should be defined as:

$$\langle \mathbf{V}_f(\mathbf{x})\rangle = \frac{1}{\tau}\int_{\tau_f} U_0(\hat{\Omega})\hat{\Omega}\,d\tau, \tag{11}$$

where τ is a small spherical volume in space, which is chosen in such a way that the radius of τ is much smaller than the characteristic length of the system so that the change of the macroscopic variable within τ can be neglected; but it is much greater than the dimensions of the pore network (the pore diameter or the inter-pore distance) so that the averaged value over τ represents the macroscopic value, and τ_f is the portion of τ occupied by the fluid which fills the void space of the tubules forming the pore network. Substitution of (10) into (11) yields the following equation:

$$\langle \mathbf{V}_f(\mathbf{x})\rangle = -\left[\frac{a^2}{8\mu\lambda}\frac{1}{\tau}\int_{\tau_f}\hat{\Omega}\hat{\Omega}\,d\tau\right]\cdot\rho_f\left(\frac{\partial \mathbf{v}_s}{\partial t} - \mathbf{g}\right)$$

$$-\frac{a^2}{8\mu\lambda}\frac{1}{\tau}\int_{\tau_f}\nabla p\cdot\hat{\Omega}\hat{\Omega}\,d\tau. \tag{12}$$

The integral over τ_f can be replaced by an integral over τ by studying Fig. 1. Let us designate the radius of the spherical volume τ by δ. The number of pores with direction vector $\hat{\Omega}$ passing through a vector area element $\delta^2\,d\omega$ is equal to $[f(\mathbf{x},\hat{\Omega})\hat{\Omega}\cdot\hat{\omega}]\delta^2\,d\omega$. The volume of the void space of a tubules within the spherical volume τ is equal to

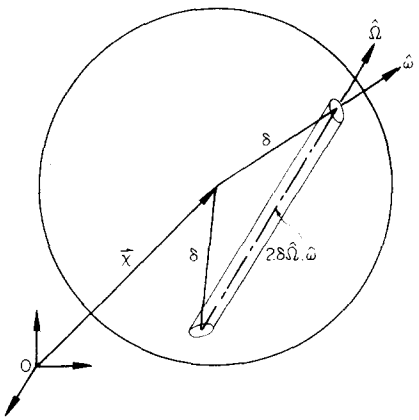

FIG. 1. Geometry of a small spherical volume over which the microscopic value is averaged to obtain the macroscopic value.

$2\pi a^2\delta\hat{\Omega}\,.\,\hat{\omega}$. Therefore, we find the following relation:

$$\int_{\tau_f} Q(\mathbf{x},\hat{\Omega})\,d\tau = \frac{1}{2}\left\{2\delta^3\oint_{\omega}\left[\int_{\Omega_{\hat{\omega}}=0}^{2\pi}\pi a^2 Q(\mathbf{x},\hat{\Omega})f(\mathbf{x},\hat{\Omega})(\hat{\Omega}\,.\,\hat{\omega})^2\,d\Omega\right]d\omega\right\}, \tag{13}$$

where the factor $\frac{1}{2}$ is introduced in correspondence with the fact that each pore is counted twice in the integral over the whole solid angle $\omega = 4\pi$. The distribution of $f(\mathbf{x},\hat{\Omega})$ in Ω has to be point symmetric. Therefore, we can write (13) as:

$$\int_{\tau_f} Q(\mathbf{x},\hat{\Omega})\,d\tau = \frac{\delta^3}{2}\left[\oint\hat{\omega}\hat{\omega}\,d\omega\right]:\left[\oint\pi a^2 Q(\mathbf{x},\hat{\Omega})f(\mathbf{x},\hat{\Omega})\hat{\Omega}\hat{\Omega}\,d\Omega\right]. \tag{14}$$

It can be easily shown that $\oint\hat{\omega}\hat{\omega}\,d\omega = (4\pi/3)\mathscr{I}$; where $\mathscr{I}$ designates the idemfactor. Thus, (14) becomes

$$\int_{\tau_f} Q(\mathbf{x},\hat{\Omega})\,d\tau = \frac{\tau}{2}\oint\pi a^2 Q(\mathbf{x},\hat{\Omega})f(\mathbf{x},\hat{\Omega})\,d\Omega. \tag{15}$$

By using the above relation, we can write (12) in the following form:

$$\langle\mathbf{V}_f(\mathbf{x})\rangle = -\left[\frac{\pi a^4}{16\mu\lambda}\oint f(\mathbf{x},\hat{\Omega})\hat{\Omega}\hat{\Omega}\,d\Omega\right]\cdot\left\{\rho_f\left(\frac{\partial\mathbf{v}_s}{\partial t}-\mathbf{g}\right)+\nabla\langle p\rangle\right\}, \tag{16}$$

where $\nabla\langle p\rangle$ is the averaged value of the pressure gradient of the fluid over the solid angle, which is

$$\nabla\langle p\rangle\,.\oint f(\mathbf{x},\hat{\Omega})\hat{\Omega}\hat{\Omega}\,d\Omega = \oint(\nabla p)f(\mathbf{x},\hat{\Omega})\,.\,\hat{\Omega}\hat{\Omega}\,d\Omega. \tag{17}$$

One may question the above definition of the average pressure which differs from the simple space averaged value of the pressure. In this case, the pressure depends on the position as well as on the direction of the axis of the tubule. In other words, p is function of $\mathbf{x}$ and $\hat{\Omega}$ and it differs from conventional pressure which does not depend on the direction. As a

consequence, we have to take the average value of a vector quantity generated from the pressure data instead of the pressure itself, which is a scalar quantity.

We may write (16) in the following form:

$$\langle \mathbf{V}_f \rangle = -\frac{\not{k}}{\mu} \cdot \left[\rho_f \left(\frac{\partial \mathbf{v}_s}{\partial t} - \mathbf{g} \right) + \nabla \langle p \rangle \right], \tag{18}$$

where $\not{k}$ is the permeability tensor

$$\not{k} = \frac{\pi a^4}{16} \oint f(\mathbf{x}, \hat{\Omega}) \hat{\Omega} \hat{\Omega} \hat{\Omega} \, \mathrm{d}\Omega. \tag{19}$$

Equation (18) is the generalized Darcy's law, in which the permeability tensor is explicitly expressed in terms of the pore matrix function $f(\mathbf{x}, \hat{\Omega})$. We note that $\not{k}$ is indeed a 2nd rank symmetric tensor as Aronofsky suggested [2].

Next, let us consider the equation of the conservation of the mass of the fluid in an arbitrary large material volume T of the porous medium. Since the volume element of the system under consideration is composed of the porous solid and the fluid moving in it, the concept of the material volume needs some explanation. If we dye a volume element of the system at a certain moment, the dyed portion of the fluid will leave the dyed portion of the solid as time lapses. Since the diameter of the pore is supposed very small, the dyed portion of the solid can be regarded as a conventional material volume of a solid continuum in a statistical sense. We define T as such hypothetical volume with smooth surface Σ, which deforms and moves with the porous solid. The flux of the fluid leaving T through a differential area element $\mathrm{d}\sigma$ on Σ can be written in two different ways (see Fig. 1)

$$\left[\int_{\Omega_{\hat{n}}=0}^{2\pi} \pi a^2 f(\mathbf{x}, \hat{\Omega}) U_0 \hat{\Omega} \cdot \hat{n} \, \mathrm{d}\Omega \right] \mathrm{d}\sigma, \tag{20a}$$

or

$$\left[- \int_{\Omega_{(-\hat{n})}=0}^{2\pi} \pi a^2 f(\mathbf{x}, \hat{\Omega}) U_0 \hat{\Omega} \cdot (-\hat{n}) \, \mathrm{d}\Omega \right] \mathrm{d}\sigma, \tag{20b}$$

where $\hat{n}$ is the outward unit normal vector on $\mathrm{d}\sigma$, and $\Omega_{(\hat{n})}$ and $\Omega_{(-\hat{n})}$ are the solid angles about $\hat{n}$ and $(-\hat{n})$, respectively. By using (20a) and (20b), we can express the flux of the fluid leaving T through $\mathrm{d}\sigma$ as

$$\left[\frac{1}{2} \oint_{\Omega} \pi a^2 f(\mathbf{x}, \hat{\Omega}) U_0 \hat{\Omega} \, \mathrm{d}\Omega \right] \cdot \hat{n} \, \mathrm{d}\sigma. \tag{21}$$

According to (15), this is equal to

$$\langle \mathbf{V}_f \rangle \cdot \hat{n} \, \mathrm{d}\sigma. \tag{22}$$

The surface integral of this flux over the closed surface Σ is equal to the time rate of decrease of the fluid mass contained in T,

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_{T_f} \rho_f \, \mathrm{d}\tau = \oint_{\Sigma} \rho_f \langle \mathbf{V}_f \rangle \cdot \hat{n} \, \mathrm{d}\sigma, \tag{23}$$

where T_f is the fluid portion of T and $\mathrm{d}/\mathrm{d}t$ designate the partial time derivative holding the surface Σ of the material volume fixed. Equation (23) can be written as

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_T \varphi\rho_f \,\mathrm{d}\tau = \oint_\Sigma \rho_f \langle \mathbf{V}_f \rangle \,.\, \hat{n}\,\mathrm{d}\sigma, \tag{24}$$

where φ is the fluid volume concentration ($\equiv$ porosity). After interchanging the order of the partial time differentiation and the volume integral on the l.h.s. of (24) by using the Reynolds transport theorem, and transforming the closed surface integral of the r.h.s. of (24) into a volume integral by using the divergence theorem, we obtain the following equation of conservation of mass for an incompressible fluid:

$$\frac{\partial \varphi}{\partial t} + \nabla \,.\, (\varphi \mathbf{v}_s) = -\nabla \,.\, \langle \mathbf{V}_f \rangle. \tag{25}$$

Equation (18) and (25) are the equations which govern the motion of the fluid through the porous solid. They involve field variables associated with the fluid as well as those of the porous solid. If the porous solid is rigid, then φ and $\mathbf{v}_s$ should be known when the problem is specified. Consequently, equations (18) and (25) supply four equations which involve four unknowns, namely, $\langle p \rangle$ and the three components of $\langle \mathbf{V}_f \rangle$. Therefore, the motion of the fluid through a rigid porous solid is completely determined by equations (18) and (25). In case of non-rigid porous solid, the motion of fluid and that of the porous solid are coupled.

3. FORMULATION OF THE STATISTICAL EQUATION OF MOTION OF THE NON-RIGID POROUS SOLID

If the porous solid is not rigid, the pore matrix function and the cross sections of the pores change as the solid deforms. Consequently, equations (18) and (25) alone are insufficient to determine the fluid motion: not only because the motion of the fluid and solid interact, but also because the permeability tensor is dependent on the state of the porous solid. Therefore, we have to determine the motion of the solid and the fluid at the same time.

In order to derive the equations governing the motion of the porous solid, we consider the equation of motion of the solid mass enclosed in an arbitrary material volume T with surface Σ, which was defined in the previous section. Let us designate the solid portion and the fluid portion of T and Σ by T_s and T_f, and Σ_s and Σ_f, respectively. The equation of motion of the solid mass within T can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{T_s} \rho_s \mathbf{v}_s \,\mathrm{d}\tau = \int_{T_s} \rho_s \mathbf{g}\,\mathrm{d}\tau + \int_{\Sigma_s} \mathbf{t}_s \,\mathrm{d}\sigma + \int_S \mathbf{t}_f \,\mathrm{d}\sigma, \tag{26}$$

where ρ_s is the density of the solid, of which the porous solid is made, $\mathbf{v}_s$ is the velocity of the solid associated with the macroscopic motion of the porous solid, $\mathbf{g}$ is the body force per unit mass and $\mathrm{d}/\mathrm{d}t$ designates the partial time derivation holding the material surface Σ fixed, and S is the surface composed of the solid–fluid interface on the wall of the tubules within T. The l.h.s. of (26) is the inertia force of the solid mass within T. The first term in the r.h.s. is the body force due to the earth's gravity, the second term is the surface traction on Σ_s due to the stress distribution within the solid medium and the

last term is the surface traction exerted on the solid mass within T by the fluid moving through it.

The surface traction acting on the solid–fluid interface has two different origins: the viscous force and the pressure force exerted by the fluid. We find from (9) that the frictional surface traction exerted by the fluid moving through a tubule of direction Ω over unit length of that tubule is equal to

$$8\pi\mu\lambda U_0(\mathbf{x}, \hat{\Omega})\hat{\Omega}. \tag{27}$$

The total frictional surface traction exerted on the solid mass within a small spherical volume τ of Fig. 1 can be found by the integral of (27) over all tubules within τ, which becomes [see the derivation of (13)]

$$\tau\left[\frac{4\mu\lambda}{a^2}\oint \pi a^2 U_0(\mathbf{x}, \hat{\Omega})\hat{\Omega} f(\mathbf{x}, \hat{\Omega})\, d\Omega\right]. \tag{28}$$

By comparing (28) with (15) after dividing (28) by τ, we find the following equation for the frictional surface traction per unit volume in space:

$$\mathbf{f}_F(\mathbf{x}) = \frac{8\mu\lambda}{a^2}\langle \mathbf{V}_f \rangle. \tag{29}$$

The net pressure force exerted on the wall of a tubule of direction $\hat{\Omega}$ over unit length of the tubule is equal to

$$\int_0^{2\pi} \hat{r} pa\, d\theta = \pi a^2 \rho_f \mathbf{g} \cdot (\mathscr{I} - \hat{\Omega}\hat{\Omega}) \tag{30}$$

where $\hat{r}$ is the outward unit normal vector on the wall of the tubule and θ is the angle measured about the axis of the tubule. The total pressure force exerted on the solid mass within small spherical volume τ can be easily found to be [see the derivation of (28) from (27)]

$$\frac{\tau}{2}\oint \pi a^2 \rho_f \mathbf{g} \cdot (\mathscr{I} - \hat{\Omega}\hat{\Omega}) f(\hat{\Omega})\, d\Omega. \tag{31}$$

From the above equation, we find the density of the surface traction due to the fluid pressure as follows:

$$\mathbf{f}_p = \varphi \rho_f \mathbf{g} \cdot (\mathscr{I} - \mathscr{H}), \tag{32}$$

where φ is the fluid volume concentration, which is equal to [see (15)]

$$\varphi = \frac{1}{2}\oint \pi a^2 f(\hat{\Omega})\, d\Omega, \tag{32a}$$

and

$$\mathscr{H} = \frac{1}{2}\oint \pi a^2 f(\hat{\Omega})\hat{\Omega}\hat{\Omega}\, d\Omega. \tag{32b}$$

By adding (32) to (29), we find the equation for the secondary body force per unit volume

created by the fluid in the porous medium as:

$$\mathbf{f}_f = \frac{8\mu\lambda}{a^2}\langle \mathbf{V}_f \rangle + \varphi\rho_f \mathbf{g} \cdot (\mathscr{I} - \mathscr{H}). \tag{33}$$

Now, the last term of the r.h.s. of (26) can be replaced by the volume integral of $\mathbf{f}_f$ over T

$$\int_S \mathbf{t}_f \, d\sigma = \int_T \left[\frac{8\mu\lambda}{a^2}\langle \mathbf{V}_f \rangle + \varphi\rho_f \mathbf{g} \cdot (\mathscr{I} - \mathscr{H}) \right] d\tau. \tag{34}$$

The porous solid under consideration is a connected solid medium (like swiss cheese) having a maze of numerous tubules whose diameter is much less than the characteristic length associated with the macroscopic deformation of the porous solid. Consequently, we assume the existence of the strain tensor which describes the macroscopic deformation of the porous solid. Since the ratio of the pore radius a to the characteristic length of the system L is supposed very small, the true deformation of the solid constituent should differ from the macroscopic value by an order of a/L or smaller and, consequently, the stress distribution has to be so. Therefore, we assume the existence of the macroscopic stress tensor and a regular relation between the macroscopic stress tensor and the macroscopic strain tensor. As mentioned in the Introduction of this paper, such relation can be found by doing stress–strain test of the porous medium in a vacuum. If we define the stress tensor $\mathscr{S}$ within the solid constituent as a second rank tensor which describes the average stress distribution within the solid constituent, then we have the relation

$$(\Delta\Sigma_s)\mathbf{t}_s = (\Delta\Sigma_s)\hat{n} \cdot \mathscr{S}, \tag{35}$$

where $\hat{n}$ is the unit normal vector on $\Delta\Sigma$. If the perturbation of the stress distribution within the porous solid due to the existence of the tubule can be neglected, then, $\mathscr{S}$ is none other than the stress tensor in a continuum out of which the porous solid is made.

Let us consider the net stress force acting on a small spherical volume τ of Fig. 1. According to the definition of the macroscopic stress tensor (35), the net force is equal to

$$\int_{\sigma_s} \mathbf{t}_s \, d\sigma = \oint_\sigma [1 - \psi(\hat{\omega})]\hat{\omega} \cdot \mathscr{S} \, d\sigma, \tag{36}$$

where σ is the surface of the spherical volume τ, σ_s is the solid portion of σ, $\psi(\hat{\omega})$ is the fluid surface area ratio ($\equiv 1 - \Delta\sigma_s/\Delta\sigma$), and $\hat{\omega}$ is the outward unit normal vector on σ. If the Taylor series expansion of $\mathscr{S}$ about the center of τ is substituted into the r.h.s. of (36), we have

$$\oint_\sigma [1 - \psi(\hat{\omega})]\hat{\omega} \cdot \mathscr{S} \, d\sigma = \left\{ \delta^2 \oint_\sigma [1 - \psi(\hat{\omega})]\hat{\omega} \, d\omega \right\} \cdot \mathscr{S}(\mathbf{x}) + \delta^3 \left\{ \oint_\sigma [1 - \psi(\hat{\omega})]\hat{\omega}\hat{\omega} \, d\omega \right\} : \nabla\mathscr{S}(\mathbf{x}) + \ldots \tag{37}$$

where $\mathbf{x}$ designates the position vector of the center of τ. The first term of the r.h.s. of (37) vanishes when the integration is carried out, because the distribution of $\psi(\hat{\omega})$ in ω has to be point symmetric. We write (37) in the following form

$$\int_{\sigma_s} \mathbf{t}_s \, d\sigma = \frac{4\pi\delta^3}{3} \mathscr{L} : \nabla\mathscr{S}, \tag{38}$$

where

$$\mathscr{L} = \frac{3}{4\pi}\oint [1-\psi(\hat{\omega})]\hat{\omega}\hat{\omega}\,\mathrm{d}\omega. \tag{38a}$$

By dividing (38) by the volume of $\tau[\equiv(4\pi/3)\delta^3]$, we find the density of the net force due to the stress in the solid

$$\mathbf{f}_s = \mathscr{L}:\nabla\mathscr{S}. \tag{39}$$

This enables us to replace the second term of the r.h.s. of (26) by a volume integral of $\mathbf{f}_s$ over T

$$\int_{\Sigma_s}\mathbf{t}_s\,\mathrm{d}\sigma = \int_T \mathscr{L}:\nabla\mathscr{S}\,\mathrm{d}\tau. \tag{40}$$

When (33) and (40) are substituted into (26) and the volume integral over T_s is replaced by the equivalent volume integral over T, the following equation results:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_T (1-\varphi)\rho_s\mathbf{v}_s\,\mathrm{d}\tau = \int_T\left\{[(1-\varphi)\rho_s\mathscr{I}+\varphi\rho_f(\mathscr{I}-\mathscr{H})]\cdot\mathbf{g}+\frac{8\mu\lambda}{a^2}\langle\mathbf{V}_f\rangle+\mathscr{L}:\nabla\mathscr{S}\right\}\mathrm{d}\tau, \tag{41}$$

which implies that

$$\frac{\partial}{\partial t}[(1-\varphi)\rho_s\mathbf{v}_s]+\nabla\cdot[(1-\varphi)\rho_s\mathbf{v}_s\mathbf{v}_s]$$

$$= [(1-\varphi)\rho_s\mathscr{I}+\varphi\rho_f(\mathscr{I}-\mathscr{H})]\cdot\mathbf{g}+\frac{8\mu\lambda}{a^2}\langle\mathbf{V}_f\rangle+\mathscr{L}:\nabla\mathscr{S}. \tag{42}$$

Now, let us consider the equation of the conservation of the solid mass in T, which is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{T_s}\rho_s\,\mathrm{d}\tau = 0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_T(1-\varphi)\rho_s\,\mathrm{d}\tau = 0, \tag{43}$$

which implies that

$$\frac{\partial}{\partial t}[(1-\varphi)\rho_s]+\nabla\cdot[(1-\varphi)\rho_s\mathbf{v}_s] = 0. \tag{44}$$

This equation of the conservation of the solid mass enable us to simplify the equation of momentum of the porous solid (42) as follows:

$$(1-\varphi)\rho_s\frac{\partial\mathbf{v}_s}{\partial t}+(1-\varphi)\rho_s\mathbf{v}_s\cdot\nabla\mathbf{v}_s = [(1-\varphi)\rho_s\mathscr{I}+\varphi\rho_f(\mathscr{I}-\mathscr{H})]\cdot\mathbf{g}+\frac{8\mu\lambda}{a^2}\langle\mathbf{V}_f\rangle+\mathscr{L}:\nabla\mathscr{S}. \tag{45}$$

As mentioned earlier, the second term of the l.h.s. of the above equation may be neglected because this term generally remains small.

Equations (18), (25), (44) and (45) are the equations governing the motion of a viscous fluid in an anisotropic non-rigid porous medium. Eight unknown quantities of the flow field are involved in this set of eight equations, which are $\langle p\rangle$, φ and the components of $\langle\mathbf{V}_f\rangle$ and $\mathbf{v}_s$. If the solid medium constituting the porous solid is compressible, then we have to add the equation of state of the solid to the governing equations and the density ρ_s

to the field variables. In order to solve this set of governing equations, the material properties involved in these equations have to be provided, namely, those for $\mathscr{k}$, $\mathscr{H}$ and $\mathscr{L}$. Here, we suppose that μ, ρ_f, ρ_s and $\mathscr{S}$ are known, which are the material properties of the constituents. Since the equation for $\mathscr{k}$ and $\mathscr{H}$ are given by (9) and (32b) in terms of the pore matrix function, respectively, we derive the constitutive equation for $\mathscr{L}$ in terms of the pore matrix function in the following section.

4. THE CONSTITUTIVE EQUATION FOR $\mathscr{L}$

In order to derive the constitutive equation for $\mathscr{L}$, which is defined by (38a), we have to find first the relation between the surface area ratio $\psi(\hat{n})$ and the pore matrix function $f(\hat{\Omega})$. Let us first find the area ratio resulting from the tubules oriented in direction $\hat{\Omega}$. The number of tubules of direction vector $\hat{\Omega}$ per unit surface area of normal vector $\hat{n}$ is equal to $\hat{\Omega}\cdot\hat{n}f(\mathbf{x},\hat{\Omega})$. The area of intersection of a tubule and the surface under consideration is equal to $\pi a^2/\hat{\Omega}\cdot\hat{n}$ provided $\hat{\Omega}\cdot\hat{n}\neq 0$. Consequently, the surface area ratio contributed by the tubules of direction $\hat{\Omega}$ is equal to

$$\mathrm{d}\psi(\hat{n}) = \pi a^2 f(\mathbf{x},\hat{\Omega}), \qquad \text{for} \quad \hat{n}\neq\hat{\Omega}. \tag{46}$$

The total surface area ratio can be found by integrating the r.h.s. of (46) over the hemispherical solid angle about $\hat{n}$. Since $f(\mathbf{x},\hat{\Omega})$ has point symmetric distribution in Ω, the surface area ratio $\psi(\hat{n})$ can be written as

$$\psi(\hat{n}) = \frac{1}{2}\pi a^2 \oint f(\mathbf{x},\hat{\Omega})\,\mathrm{d}\Omega. \tag{47}$$

According to (15), the r.h.s. of (52) is equal to the porosity. Thus, we have

$$\psi(\hat{n}) = \varphi. \tag{48}$$

In deriving the above result, we included the tubules of direction perpendicular to $\hat{n}$, when the integral of the r.h.s. of (47) is carried out. This can be justified by considering the following case. Let us consider a porous medium having numerous parallel tubules only in direction $\hat{x}$ (see Fig. 2). If we designate the number density of these tubules on the plane $x = \text{const.}$ by N, the surface area ratio on this plane is equal to $\pi a^2 N$, which is none other than the porosity. The surface area ratio on a plane of $y = \text{const.}$ is given by the equation (see Fig. 2)

$$\psi(\hat{y}) = 2N\int_{-a}^{a}\sqrt{(a^2-y^2)}\,\mathrm{d}y, \tag{49}$$

which is equal to

$$\psi(\hat{y}) = 2Na^2\int_0^{\pi}\sin^2\theta\,\mathrm{d}\theta = N\pi a^2. \tag{50}$$

This proves that $\psi(\hat{x}) = \psi(\hat{y}) = \varphi$, and consequently, the validity of (48) is proved.

Substitution of (50) into (38a) yields the following constitutive equation for $\mathscr{L}$:

$$\mathscr{L} = (1-\varphi)\mathscr{I}. \tag{51}$$

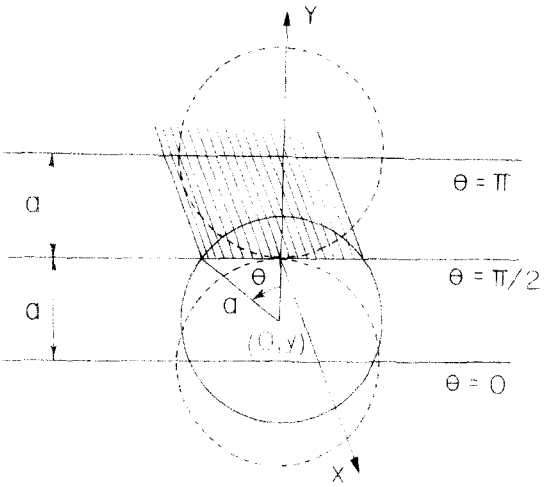

FIG. 2. Geometry of a cross section of the porous medium having tubules oriented in x-direction only.

This theoretically completes the derivation of the constitutive equations for the material properties of the porous medium, which depend only on the structures of pores. In case of a non-rigid porous media, the fluid volume ratio may change due to the change of the pore radius or the change of the pore matrix function. The description of the structure of the porous medium under consideration can be done by any combination of two out of the three variables; φ, a, $f(\hat{\Omega})$. Since a is not a macroscopic variable, we prefer to use φ and $f(\hat{\Omega})$ as the state variables of the structure of the porous medium. From (32a), we find that

$$a^2 = \frac{(2/\pi)\varphi}{\oint f(\mathbf{x}, \hat{\Omega})\, d\Omega}. \tag{52}$$

By using the above equation, we can replace a appearing in the equations governing the motion of the system.

In general, it is rather difficult to solve the set of governing equations derived in the previous sections. One of the sources of the difficulties is the change of the pore matrix function $f(\mathbf{x}, \hat{\Omega})$ as the porous solid undergoes a deformation, which causes the change of the material properties such as $\mathscr{K}$, $\mathscr{H}$ and $\mathscr{L}$. However, there are certain kinds of porous media involved in the biological system, which seldom undergo large macroscopic deformation while the change of internal structure is significant. In such cases, the change of the pore matrix function due to the reorientation of medium can be neglected, but the significant change of the fluid volume concentration is possible due to the change of the pore radius.

5. APPLICATION TO POROUS MEDIA IN THE BIOLOGICAL SYSTEM

Let us consider how those results of the previous sections can be applied to the fluid movement in tissues of a living animal. As we mentioned, the change of pore matrix function due to the change of orientation in such a system can be neglected, because it seldom

undergoes large macroscopic deformation, but the radius of the pore can vary significantly because of the swelling or shrinking of the tissue cells. We introduce the swelling (or shrinking) of the solid constituent of the porous medium as an additional property. We assume that the solid can absorb fluid, but the macroscopic motion of the fluid can take place only through the tubular passage of pores. In other words, the fluid absorbed into the solid are more or less frozen-in into the solid as far as the macroscopic motion of the solid is concerned, but the fluid content in the solid indirectly affect the macroscopic motion of the fluid by changing the diameter of the tubules. For the sake of simplification, we shall assume that the rate of absorption of the fluid by the solid is very small. We further assume that the system undergoes such an infinitesimal macroscopic deformation that any change of the pore matrix function can be ignored and that all terms which are products of two infinitesimal changes associated with the small macroscopic deformation of the medium can be neglected.

Since the rate of absorption is supposedly very small, the rate of change of the pore radius is also small. Consequently, the velocity profile in the tubules should not be affected significantly. From equation (18) and (52) we find the following equation after neglecting terms which are products of two or more infinitesimal changes:

$$\langle \mathbf{V}_f \rangle = -\frac{\varphi^2}{\mu} \mathbb{k}_0 \cdot \left[\rho_f \left(\frac{\partial \mathbf{v}_s}{\partial t} - \mathbf{g} \right) + \nabla \langle p \rangle \right], \tag{53}$$

where φ is the ratio of the volume of the tubules to that of the porous medium and $\mathbb{k}_0$ is the permeability tensor

$$\mathbb{k}_0 = \frac{1}{4\lambda\pi} \frac{\oint f(\hat{\Omega}) \hat{\Omega}\hat{\Omega} \, d\Omega}{[\oint f(\hat{\Omega}) \, d\Omega]^2}. \tag{54}$$

Since φ is the volume occupied by the tubules in unit volume of the medium, $(1-\varphi)$ is the volume occupied by the solid and the absorbed fluid in unit volume of the medium. Let us introduce the fluid content C in the solid as the ratio of the volume of the absorbed fluid to the total volume of the solid and the absorbed fluid. Then, the total volume of the fluid per unit volume of the medium is equal to $[\varphi+(1-\varphi)C]$ and the pure solid volume per unit volume of the medium is $(1-\varphi)(1-C)$.

It was assumed that the macroscopic motion of the fluid takes place only through the tubular passage of pores and, consequently, the net inflow of the fluid per unit material volume of the medium remain to be $-\nabla \cdot \langle \mathbf{V}_f \rangle$. Therefore, the equation of conservation of mass for the fluid can be obtained from equation (25) after replacing φ by $\varphi+(1-\varphi)C$,

$$\nabla \cdot \{ \langle \mathbf{V}_f \rangle + [\varphi+(1-\varphi)C] \mathbf{v}_s \} = -\frac{\partial}{\partial t} [\varphi+(1-\varphi)C]. \tag{55}$$

The equation of motion of the porous solid and trapped fluid in it can be found from (45), (51) and (52) by replacing ρ_s by $[(1-C)\rho_s + C\rho_f]$ after substituting (53) into (45). By doing so, we find

$$m \cdot \frac{\partial \mathbf{v}_s}{\partial t} = m \cdot \mathbf{g} + \varphi[\rho_f(\mathscr{I} - \varphi \mathscr{H}_0) \cdot \mathbf{g} - \mathscr{H}_0 \cdot \nabla \langle p \rangle] + (1-\varphi) \nabla \cdot \mathscr{S}, \tag{56}$$

where

$$\mathscr{M} = (1-\varphi)[(1-C)\rho_s + C\rho_f]\mathscr{I} + \varphi\rho_f\mathscr{H}_0, \tag{57}$$

$$\mathscr{H}_0 = \frac{\oint f(\hat{\Omega})\hat{\Omega}\hat{\Omega}\,\mathrm{d}\Omega}{\oint f(\hat{\Omega})\,\mathrm{d}\Omega}. \tag{58}$$

The equation of the conservation of mass for the solid can be found from (44) by replacing ρ_s by $(1-C)\rho_s$

$$\frac{\partial}{\partial t}[(1-\varphi)(1-C)\rho_s] + \nabla\cdot[(1-\varphi)(1-C)\rho_s\mathbf{v}_s] = 0. \tag{59}$$

$\mathscr{k}_0$ and $\mathscr{H}_0$ are material constants, which depend only on $f(\hat{\Omega})$, as can be seen clearly from (54) and (58). If $f(\hat{\Omega})$ is given, then we can determine these material constants. However, in many cases it is difficult to directly determine $f(\hat{\Omega})$ experimentally. Therefore, it is most likely that these material constants would have to be determined by some experimental methods using one or more relations presented in this study. Once they are known, equations (53), (55), (56) and (59) supply a set of eight equations which involve nine unknowns; $\langle p\rangle$, φ, C and those components of $\langle\mathbf{V}_f\rangle$ and $\mathbf{v}_s$. To make the set of equations complete, one more equation has to be supplied. It is obvious that this additional equation should be the equation governing the fluid content C in the solid. Such an equation should not be derived on a simple basis of the mechanical equilibrium, because the control of the fluid content in the tissue cell most likely involves more complicated physiological process. Therefore, we end the present study in presenting the physical nature of the problem by deriving a set of equations governing the fluid movement through the non-rigid porous medium, which can be solved provided the equation of fluid content is known or a certain change of the fluid content is assumed.

6. CONCLUSION

A set of equations governing the macroscopic motion of the constituents (the porous solid and the fluid within the pores) of a porous medium is derived for a mathematical model of a porous medium involving a network of densely distributed circular cylindrical tubules of very small diameters. These equations are applied to a biological system in which the effect of swelling (or shrinking) of the porous solid by the absorption of the fluid was taken into consideration. The generalized Darcy's law (18) shows explicitly the permeability tensor as a function of the pore structure (the pore matrix function). It is also clearly shown by (19) that the permeability tensor is symmetric as Aronofsky suggested. The equation of the macroscopic motion of the porous solid shows the macroscopic material properties of the solid portion constituents as a function of the pore structure and the material properties of constituents. For example, we note in equation (57) that the effective density is a symmetric tensor of rank two instead of a scalar. This is rather natural because the macroscopic response of the porous solid to a body force such as the inertia force caused by the acceleration should have a preferred direction

because the resistance to the motion of the fluid within the anisotropically distributed pores has different values in different directions. We also notice the secondary body force exerted on the solid medium created by the weight of the fluid and the fluid pressure gradient. It is interesting to notice in the final form of the governing equations that field variables and constitutive variables are well distinguished from each other. Of course, the form of the governing equations and constitutive equations derived in this paper should be applied only to the specific model of the porous medium employed in this paper. Nevertheless, we speculate that the form of the governing equation may be applied to a more general model as long as the porous solid is a connected medium, because the detailed pore structure enters into the picture only through the constitutive equations.

Acknowledgments—This work is supported partially by the National Science Foundation under Grant No. GK-1415; partially by the San Diego County Heart Association under a Fellowship granted in the name of the late Marjorie Siner; and, Grant-in-aid from American Heart Association to the Department of Physiology, University of Arizona.

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(*Received* 27 *May* 1969; *revised* 20 *January* 1970)

Абстракт—Исследуется движение вязкой жидкости сквозь анизотропное, не-жесткое пористое твердое тело. Рассматриваемое в этой работе пористое тело обсуждается как твердая среда, заключающая оуень густую конечную сеть трубок, диаметр которых значителрно меньший по сравнению с характеристической длиной системы течения. С кинематической точки зрения, такая среда должна вести себя как континуум, т.е. деформированное состояние можно описапь, надлежащим способом, макроскопически. С динамиуеской точки зрения, такой тип среды должен отличаться от настоящей твердой сплошной среды, так как при деформации пористого тела должна быть учтена одна добавочная массовая сила, добавленная к ему весу /поверхностная сила сцепления, действующая в виду движения вязкой жидкости сквозь среду/. Единственной массовой силой в настоящей твердой сплошной среде является ее собственный вес. Даются определяющие уравнения макроскопического движения чотока вязкой жидкости сквозр пористую среду такого же типа, путем усреднения движения жидкости сквозь отдельные элементы поров в малом объеме пористой среды. Обсуждается физическая интерпретация полученных уравнений, а также возможные применения к задачам, касающимся течения крови в сети капилярных кровеноносных сосудов цк задачам движения весьма зчеистых жидкостей сквозь структуру пространства с трещинами.